On teaching sets of k-threshold functions

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Abstract

Let f be a $\{0,1\}$-valued function over an integer d-dimensional cube $\{0,1,\ldots,n-1\}^d$, for $n \geq 2$ and $d \geq 1$. The function f is called threshold if there exists a hyperplane which separates 0-valued points from 1-valued points. Let C be a class of functions and $f \in C$. A point x is essential for the function f with respect to C if there exists a function $g \in C$ such that x is a unique point on which f differs from g. A set of points X is called teaching for the function f with respect to C if no function in $C \setminus \{f\}$ agrees with f on X. It is known that any threshold function has a unique minimal teaching set, which coincides with the set of its essential points. In this paper we study teaching sets of k-threshold functions, i.e. functions that can be represented as a conjunction of k threshold functions. We reveal a connection between essential points of k threshold functions and essential points of the corresponding k-threshold function. We note that, in general, a k-threshold function is not specified by its essential points and can have more than one minimal teaching set. We show that for $d = 2$ the number of minimal teaching sets for a 2-threshold function can grow as $\Omega(n^2)$. We also consider the class of polytopes with vertices in the d-dimensional cube. Each polytope from this class can be defined by a k-threshold function for some k. In terms of k-threshold functions we prove that a polytope with vertices in the d-dimensional cube has a unique minimal teaching set which is equal to the set of its essential points. For $d = 2$ we describe structure of the minimal teaching set of a polytope and show that cardinality of this set is either $\Theta(n^2)$ or $O(n)$ and depends on the perimeter and the minimum angle of the polytope.

Keywords: machine learning, threshold function, essential point, teaching set, learning complexity, k-threshold function

1. Introduction

Let n and d be integers such that $n \geq 2$ and $d \geq 1$ and let E_n^d denote a d-dimensional cube $\{0,1,\ldots,n-1\}^d$. A function f that maps E_n^d to $\{0,1\}$ is *threshold*, if there exist real numbers $a_0, a_1, \ldots, a_d$ such that

$$M_1(f) = \left\{ x \in E_n^d : \sum_{j=1}^{d} a_j x_j \leq a_0 \right\},$$

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where $M_\nu(f)$ is the set of points $x \in E_n^d$ for which $f(x) = \nu$. The inequality $\sum\limits_{j=1}^{d} a_j x_j \le a_0$ is called *threshold*. We denote by $\mathfrak{T}(d,n)$ the class of all threshold functions over E_n^d.

Let k be a natural number. A function f that maps E_n^d to $\{0,1\}$ is called *k-threshold* if there exist real numbers $a_{10}, a_{11}, \dots, a_{kd}$ such that

$$M_1(f) = \left\{ x \in E_n^d : \sum_{j=1}^{d} a_{ij} x_j \le a_{i0}, \quad i = 1, \dots, k \right\}. \tag{1}$$

The system of inequalities $\sum\limits_{j=1}^{d} a_{ij} x_j \le a_{i0}, \quad i = 1, \dots, k$ is called *threshold* and *defines* the k-threshold function f. Let $\mathfrak{T}(d,n,k)$ be the class of k-threshold functions over E_n^d. By definition $\mathfrak{T}(d,n,1) = \mathfrak{T}(d,n)$. Note that a k-threshold function is also a j-threshold function for $j > k$. Denote by $\mathfrak{T}(d,n,*)$ the class of all k-threshold functions over E_n^d for all natural k, that is $\mathfrak{T}(d,n,*) = \bigcup\limits_{k \ge 1} \mathfrak{T}(d,n,k)$.

For any k-threshold function f there exist threshold functions $f_1, \dots, f_k$ such that

$$f(x) = f_1(x) \wedge \dots \wedge f_k(x),$$

where "$\wedge$" denotes the usual logical conjunction. We will say that f is *defined* by $f_1, \dots, f_k$ and $\{f_1, \dots, f_k\}$ is *defining set* for f.

A convex hull of a set of points $X \subseteq \mathbb{R}^d$ is denoted by $\mathrm{Conv}(X)$. For a function $f : E_n^d \to \{0,1\}$ we denote by $P(f)$ the convex hull of $M_1(f)$, that is $P(f) = \mathrm{Conv}(M_1(f))$. For any polytope P with vertices in E_n^d there exists a unique k-threshold function f, such that $P = P(f)$. Therefore there is one-to-one correspondence between functions in the class $\mathfrak{T}(d,n,*)$ and polytopes with vertices in E_n^d, and we can say that $\mathfrak{T}(d,n,*)$ is a *class of polytopes with vertices in E_n^d*.

In [1] Angluin considered a model of concept learning with membership queries. In this model a *domain* X and a *concept class* $\mathcal{S} \subseteq 2^X$ are known to both the *learner* (or *learning algorithm*) and the *oracle*. The goal of the learner is to identify an unknown *target concept* $S_T \in \mathcal{S}$ that has been fixed by the oracle. To this end, the learner may ask the oracle membership queries "does an element x belong to S_T?", to which the oracle returns "yes" or "no". The learning complexity of a learning algorithm $\mathcal{A}$ with respect to a concept class $\mathcal{S}$ is the minimum number of membership queries sufficient for $\mathcal{A}$ to identify any concept in $\mathcal{S}$. The *learning complexity of a concept class* $\mathcal{S}$ is defined as the minimum learning complexity of a learning algorithm with respect to $\mathcal{S}$ over all learning algorithms which learn $\mathcal{S}$ using membership queries.

In terms of Angluin's model, a $\{0,1\}$-valued functions over E_n^d can be considered as a characteristic functions of concepts. Here E_n^d is the domain and a function $f : E_n^d \to \{0,1\}$ defines a concept $M_1(f)$. Concept learning with membership queries for classes of threshold functions, k-threshold functions, and polytopes with vertices in E_n^d corresponds to the problem of identifying geometric objects in E_n^d with certain properties.

From results of [2] and [3] it follows that the learning complexity of the class of threshold functions $\mathfrak{T}(d,n)$ is $O\left(\frac{\log_2^{d-1} n}{\log \log_2 n}\right)$. In [4] Maass and Bultman studied learning complexity of the class k-$HALFSPACES^2_{n,p}$, where $0 < p \le \frac{\pi}{2}$. The class k-$HALFSPACES^2_{n,p}$ is the subclass of k-threshold functions over E_n^2 with restrictions

that for any f in this subclass $P(f)$ has edges with length at least $16 \cdot \left\lceil \frac{1}{p} \right\rceil$ and an angle α between a pair of adjacent edges satisfies $p \leq \alpha \leq \pi - p$. The learning algorithm proposed in [4] for identification a function f in k-$HALFSPACES^2_{n,p}$ requires a vertex of the polygon $P(f)$ as input and uses $O(k(\frac{1}{p} + \log n))$ membership queries.

Let $\mathcal{C}$ be a class of $\{0,1\}$-valued functions over the domain X and $f \in \mathcal{C}$. A *teaching set* of a function f with respect to $\mathcal{C}$ is a set of points $T \subseteq X$ such that the only function in $\mathcal{C}$ which agrees with f on T is f itself. A teaching set T is *minimal* if no of its proper subset is teaching for f. Note that a teaching set of a function $f \in \mathfrak{T}(d,n,k)$ with respect to $\mathfrak{T}(d,n,*)$ is a teaching set with respect to $\mathfrak{T}(d,n,k)$. A point $x \in X$ is called *essential* for a function $f \in \mathcal{C}$ with respect to $\mathcal{C}$ if there exists a function $g \in \mathcal{C}$ such that $f(x) \neq g(x)$ and f agrees with g on $X \setminus \{x\}$. Let us denote the set of essential points of a function f with respect to a class $\mathcal{C}$ by $S(f,\mathcal{C})$ or by $S(f)$ when $\mathcal{C}$ is clear. Let $S_\nu(f) = S(f) \cap M_\nu(f)$. By $J(f,C)$ we denote the number of minimal teaching sets of a function f with respect to a class C and by $\sigma(f,\mathcal{C})$ the minimum cardinality of a teaching set of f with respect to $\mathcal{C}$. The *teaching dimension* of a class $\mathcal{C}$ is defined as

$$\sigma(\mathcal{C}) = \max_{f \in \mathcal{C}} \sigma(f,\mathcal{C}).$$

The main aim of a learning algorithm with membership queries is to find any teaching set of a target function f with respect to a concept class $\mathcal{C}$. The algorithm succeeds if it asked queries in all points of some teaching set of the function. Therefore the teaching dimension of the class $\mathcal{C}$ is a lower bound on the learning complexity of this class.

It is known (see, for example, [5] and [6]), that the set of essential points of a threshold function is a teaching set of this function. Together with the simple observation that any teaching set of a function should contains all its essential points, this imply that any threshold function have a unique minimal teaching set, that is $J(f,\mathfrak{T}(d,n)) = 1$. In addition, it follows from [3, 6, 7, 8] that for any fixed $d \geq 2$

$$\sigma(\mathfrak{T}(d,n)) = \Theta(\log_2^{d-2} n) \quad (n \to \infty).$$

In this paper we study combinatorial and structural properties of teaching sets of k-threshold functions for $k \geq 2$. In particular, we show that 2-threshold functions from $\mathfrak{T}(2,n,2)$, in contrast with threshold functions, can have more than one minimal teaching set with respect to $\mathfrak{T}(2,n,2)$. Moreover, we construct a sequence of functions from $\mathfrak{T}(2,n,2)$ for which number of minimal teaching sets grows as $\Omega(n^2)$. On the other hand, we show that any k-threshold function f (or a polytope with vertices in E_n^d) has a unique minimal teaching set with respect to $\mathfrak{T}(d,n,*)$ coinciding with the set of essential points of f with respect to $\mathfrak{T}(d,n,*)$. In addition, we give a general description of minimal teaching sets of such functions. For functions in $\mathfrak{T}(2,n,*)$ we refine the given structure and derive a bound on the cardinality of the minimal teaching sets.

The organization of the paper is as follows. In Section 2, we consider essential points of an k-fold conjunction of an arbitrary $\{0,1\}$-valued functions $f_1,\ldots,f_k$ and their connection with essential points of these functions. In the beginning of Section 3 we show that in general a k-threshold function can have more than one minimal teaching set. The main result of Subsection 3.1 (Theorem 8) states that a minimal teaching set of a k-threshold function with respect to $\mathfrak{T}(d,n,*)$ is unique and coincides with $S(f,\mathfrak{T}(d,n,*))$. The structure of $S(f,\mathfrak{T}(d,n,*))$ is given as well. In Subsection 3.2 we consider the class

$\mathfrak{T}(2,n,*)$ and for a function f in the class we prove an upper bound on the cardinality of $S(f,\mathfrak{T}(2,n,*))$. Finally, in Subsection 3.3 we consider functions in $\mathfrak{T}(2,n,2)$ with special properties and show that each of these functions has a minimal teaching set with cardinality at most 9 and there are functions with $\Omega(n^2)$ minimal teaching sets with respect to $\mathfrak{T}(2,n,2)$.

2. The set of essential points of a $\{0,1\}$-valued functions conjunction

Since a k-threshold function is a conjunction of k threshold functions, it is interesting to investigate connection between essential points of threshold functions $f_1,\dots,f_k$ and essential points of their conjunction. In this section we prove several propositions that establish this relationship. For a natural $k>1$ and a class $\mathcal{C}$ of $\{0,1\}$-valued functions we denote by $\mathcal{C}^k$ the class of functions which can be presented as conjunction of k functions from $\mathcal{C}$.

Proposition 1. *Let $\mathcal{C}$ be a class of $\{0,1\}$-valued functions over a domain X and $f_1,\dots,f_k\in\mathcal{C}$. Then for the function $f=f_1\wedge\dots\wedge f_k$ the following inclusions hold:*

$$S_1(f_i,\mathcal{C})\cap M_1(f)\subseteq S_1(f,\mathcal{C}^k)\qquad (i=1,\dots,k).$$

Proof. Let $x\in S_1(f_i,\mathcal{C})\cap M_1(f)$ for some $i\in\{1,\dots,k\}$. Since x is an essential point of f_i and $f_i(x)=1$, there exists a function $f_i'\in\mathcal{C}$ which differs from f_i in the unique point x. Denote by f' the conjunction $f_1\wedge\ldots\wedge f_{i-1}\wedge f_i'\wedge f_{i+1}\wedge\ldots\wedge f_k$. The function f' belongs to the class $\mathcal{C}^k$ and differs from f in the unique point x, namely $f'(x)=0\neq f(x)$. It means that x is essential for f, i.e. $x\in S_1(f,\mathcal{C}^k)$. ∎

Proposition 2. *Let $\mathcal{C}$ be a class of $\{0,1\}$-valued functions over a domain X and $f_1,\dots,f_k\in\mathcal{C}$. Then for the function $f=f_1\wedge\dots\wedge f_k$ the following inclusions hold:*

$$S_0(f_i,\mathcal{C})\cap\bigcap_{j\neq i}M_1(f_j)\subseteq S_0(f,\mathcal{C}^k)\qquad (i=1,\dots,k).$$

Proof. Let $x\in S_0(f_i,\mathcal{C})\cap\bigcap\limits_{j\neq i}M_1(f_j)$ for some $i\in\{1,\dots,k\}$. Since $x\in S_0(f_i)$, there exists a function $f_i'\in\mathcal{C}$ such that $f_i'(x)=1$ and $f_i'(y)=f_i(y)$ for every $y\in X\setminus\{x\}$. Denote by f' the conjunction $f_1\wedge\ldots\wedge f_{i-1}\wedge f_i'\wedge f_{i+1}\wedge\ldots\wedge f_k$. The function f' belongs to the class $\mathcal{C}^k$ and, since $x\in\bigcap\limits_{j\neq i}M_1(f_j)$, it differs from f in the unique point x, namely $f'(x)=1\neq f(x)$. Therefore x is essential for f and $x\in S_0(f,\mathcal{C}^k)$. ∎

Proposition 3. *Let $\mathcal{C}$ be a class of $\{0,1\}$-valued functions over a domain X and $f\in\mathcal{C}^k$. If there exists a unique set $f_1,\dots,f_k\in\mathcal{C}$ such that $f=f_1\wedge\dots\wedge f_k$, then*

$$S(f_i,\mathcal{C})\subseteq\bigcap_{j\neq i}M_1(f_j)\qquad (i=1,\dots,k).$$

Proof. Suppose to the contrary that there exists $x\in X$ such that $x\in S(f_i,\mathcal{C})$ and $f_j(x)=0$ for some distinct indices $i,j\in\{1,\dots,k\}$. It means that $f(x)=0$. Since x is essential for f_i, there exists a function $f_i'\in\mathcal{C}$ which differs from f_i in the unique point x. Clearly, $f_1\wedge\ldots\wedge f_{i-1}\wedge f_i'\wedge f_{i+1}\wedge\ldots\wedge f_k=f$, which contradicts the uniqueness of the set $\{f_1,\dots,f_k\}$. ∎

Corollary 4. *Let $\mathcal{C}$ be a class of $\{0,1\}$-valued functions over a domain X and $f \in \mathcal{C}^k$. If there exists a unique set $f_1, \dots, f_k \in \mathcal{C}$ such that $f = f_1 \wedge \dots \wedge f_k$ then*

$$\bigcup_{i=1}^{k} S_\nu(f_i, \mathcal{C}) \subseteq S_\nu(f, \mathcal{C}^k) \qquad (\nu = 0, 1).$$

Proof. Since the function f satisfies the conditions of Proposition 3,

$$S_1(f_i, \mathcal{C}) \subseteq M_1(f) \qquad (i = 1, \dots, k)$$

and

$$S_0(f_i, \mathcal{C}) \subseteq \bigcap_{j \neq i} M_1(f_j) \qquad (i = 1, \dots, k).$$

By Propositions 1 and 2 we get

$$\bigcup_{i=1}^{k} S_\nu(f_i, \mathcal{C}) \subseteq S_\nu(f, \mathcal{C}^k).$$

■

3. Teaching sets of k-threshold functions

Recall that the minimal teaching set of a threshold function is unique and equal to the set of its essential points. The situation becomes different for k-threshold functions when $k \geq 2$. We illustrate this difference in the following example.

Example 5. *Let f be a function from $\mathfrak{T}(2,4,2)$ with*

$$M_1(f) = \{(1,2), (1,3), (2,2), (2,3)\}.$$

The set of essential points $S(f)$ is

$$\{(1,1), (1,2), (2,1), (2,2), (0,3), (3,3)\}.$$

This set is not a teaching set because there exists a function $g \in \mathfrak{T}(2,4,2)$ with $M_1(g) = \{(1,2), (2,2)\}$, which agrees with f on $S(f)$ (see Fig. 1). Though if we add any of the two points $(1,3)$ or $(2,3)$ to $S(f)$, then we get a minimal teaching set of the function f (see Fig. 2) with respect to $\mathfrak{T}(2,4,2)$, and therefore $J(f, \mathfrak{T}(2,4,2)) \geq 2$.

3.1. Teaching sets for functions in $\mathfrak{T}(d,n,*)$

In this section we prove that for $k \geq 2$ and $d \geq 2$ the teaching dimension of $\mathfrak{T}(d,n,k)$ is n^d. Then we consider the class $\mathfrak{T}(d,n,*)$ and show that for a function $f \in \mathfrak{T}(d,n,*)$ the set of its essential points with respect to $\mathfrak{T}(d,n,*)$ is also a teaching set, and therefore it is a unique minimal teaching set of f with respect to $\mathfrak{T}(d,n,*)$.

Lemma 6. *Let $f : E_n^d \to \{0,1\}$ be a function such that $1 \leq |\mathrm{Vert}(P(f))| \leq 2$ and $P(f) \cap M_0(f) = \emptyset$. Then $f \in \mathfrak{T}(d,n,k)$ for any $k \geq 2$.*

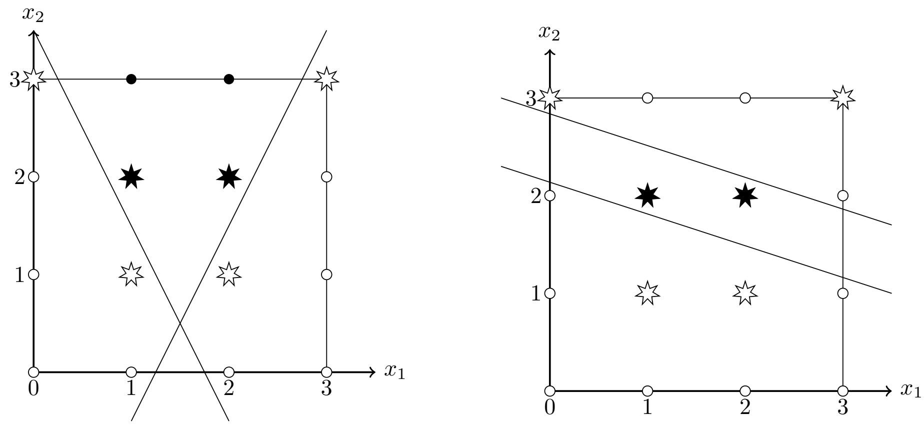

Figure 1: The stars denote the essential points. The black elements denote the points from $M_1(f)$. The empty elements denote the points from $M_0(f)$. The functions f (left plot) and g (right plot) agree on $S(f)$.

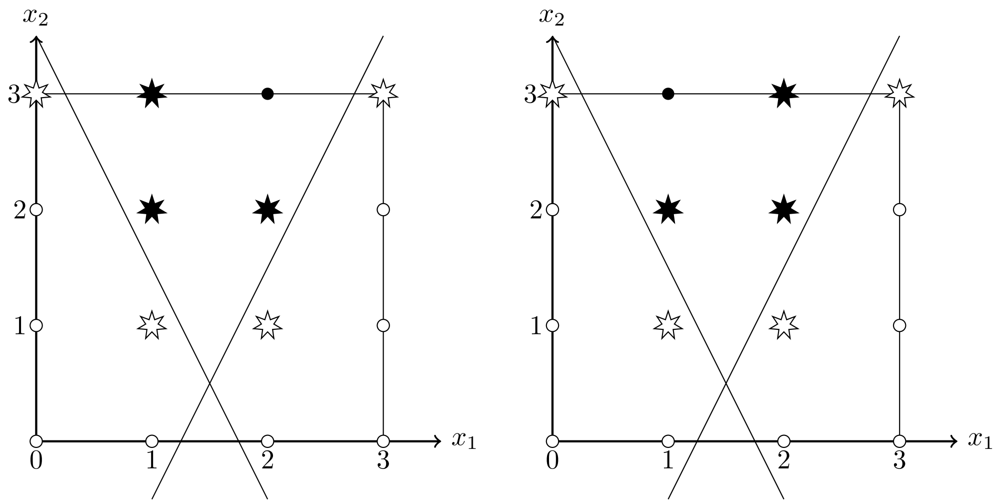

Figure 2: The stars denote the points of the minimal teaching sets $S(f) \cup \{1, 3\}$ (left plot) and $S(f) \cup \{2, 3\}$ (right plot).

Proof. It is sufficient to show that f is a 2-threshold function. Let x and y be the two vertices of $P(f)$. Note that if $|M_1(f)| = 1$, then $x = y$.

Clearly, it is possible to choose two parallel hyperplanes H' and H'' sufficiently close to each other such that $E_n^d \cap H' = \{x\}$, $E_n^d \cap H'' = \{y\}$, and there are no points between H' and H'' in $E_n^d \backslash M_1(f)$. These hyperplanes can be used to define a 2-threshold function, that coincides with f. ■

In [1] it was established that the teaching dimension of a class containing the empty set and N singleton sets is at least N. This result and Lemma 6 give us the teaching dimension for $\mathfrak{T}(d, n, k)$, where $k \geq 2$:

Corollary 7. *$\sigma(\mathfrak{T}(d, n, k)) = n^d$ for every $k \geq 2$.*

For a polytope P denote by $\mathrm{Vert}(P)$ the set of vertices of P, by $B(P)$ the set of integer points on the border of P and by $\mathrm{Int}(P)$ the set of internal integer points of P. For $f \in \mathfrak{T}(d, n, *)$ denote by $D(f)$ the set $\{x \in M_0(f) : \mathrm{Conv}(P(f) \cup \{x\}) \cap M_0(f) = \{x\}\}$.

Theorem 8. *Let $f \in \mathfrak{T}(d, n, *)$, $d \geq 2$, $n \geq 2$. Then*

$$S(f, \mathfrak{T}(d, n, *)) = \begin{cases} E_n^d, & M_1(f) = \emptyset; \\ \mathrm{Vert}(P(f)) \cup D(f), & M_1(f) \neq \emptyset; \end{cases}$$

and $S(f, \mathfrak{T}(d, n, *))$ is a unique minimal teaching set of f.

Proof. If $M_1(f) = \emptyset$, then $f \equiv 0$, and therefore $S(f) = E_n^d$. Clearly, in this case $S(f)$ is a unique minimal teaching set for f.

Now let $M_1(f) \neq \emptyset$. We split the proof of this case into two parts. At first we show that all points from $\mathrm{Vert}(P(f)) \cup D(f)$ are essential, and then we prove that this set is a unique minimal teaching set.

1. Let $f' : E_n^d \to \{0, 1\}$ be a function which differs from f in a unique point $x \in \mathrm{Vert}(P(f))$. Obviously $P(f') \cap M_0(f') = \emptyset$ and f' belongs to $\mathfrak{T}(d, n, *)$. Therefore x is essential for f with respect to $\mathfrak{T}(d, n, *)$. Now let $f' : E_n^d \to \{0, 1\}$ be a function which differs from f in a unique point $x \in D(f)$. The choice of x implies that the function f' belongs to $\mathfrak{T}(d, n, *)$ and hence x is essential point of f with respect to $\mathfrak{T}(d, n, *)$.
2. Since f belongs to the class $\mathfrak{T}(d, n, *)$, knowing values of the function in $\mathrm{Vert}(P(f))$ is sufficient to recover f in $M_1(f)$. Further, for every point $x \in M_0(f)$ such that $|\mathrm{Conv}(P(f) \cup \{x\}) \cap M_0(f)| > 1$ the set $\mathrm{Conv}(P(f) \cup \{x\}) \cap M_0(f)$ necessarily contains at least one point from $D(f)$. Therefore, to recover f in $M_0(f)$ it is sufficient to know the function values in points from $D(f)$ and $\mathrm{Vert}(P(f))$. This leads us to a conclusion that $\mathrm{Vert}(P(f)) \cup D(f)$ is a teaching set. Moreover, since all points in this set are essential and any teaching set contains all essential points, we conclude that $\mathrm{Vert}(P(f)) \cup D(f)$ is a unique minimal teaching set and coincides with $S(f)$.

■

Lemma 9. *Let $f \in \mathfrak{T}(d, n, k), d \geq 2, k \geq 2$ and $M_1(f) = \{x'\}$. Then*

$$S(f, \mathfrak{T}(d, n, k)) = \{x'\} \cup \{x \in E_n^d : \mathrm{GCD}(|x_1 - x_1'|, \dots, |x_d - x_d'|) = 1\},$$

and $S(f,\mathfrak{T}(d,n,k))$ is a unique teaching set of f with respect to $\mathfrak{T}(d,n,k)$ and

$$|S(f,\mathfrak{T}(d,n,k))| = \Theta(n^d).$$

Proof.

Let $S = \{x \in E_n^d : \mathrm{GCD}(|x_1 - x'_1|, \ldots, |x_d - x'_d|) = 1\}$. For any $x \in S$ the segment $x'x$ does not contain other points from E_n^d except x and x', that is $x'x \cap E_n^d = \{x', x\}$. Then, according to Lemma 6, a function $g: E_n^d \to \{0,1\}$ with $M_1(g) = \{x', x\}$ belongs to the class $\mathfrak{T}(d,n,k)$ for any $k \geq 2$. Since x distinguishes g from f, it is an essential point for the both functions. Therefore all points of S are essential for f. On the other hand, $S \cup \{x'\}$ is a teaching set for f because for any point $y \in E_n^d \setminus \{S \cup \{x'\}\}$ there exists a point $y' \in S$ such that y, y', x' are collinear and y' is between y and x'.

Let $\varphi(i)$ be the Euler totient function. It is well known (see, for example, [9]) that

$$\sum_{i \leq n} \varphi(i) = \frac{3}{\pi^2} n^2 + O(n \ln n).$$

Using this formula we can get a lower bound on the cardinality of the minimal teaching set:

$$|S \cup \{x'\}| = \left|\{x = (x_1, \ldots, x_d) \in E_n^d : \mathrm{GCD}(|x_1 - x'_1|, \ldots, |x_d - x'_d|) = 1\}\right| + 1 \geq$$

$$\geq \left|\{x = (x_1, \ldots, x_d) \in E_n^d : x_3 = \cdots = x_d = 0, \mathrm{GCD}(|x_1 - x'_1|, |x_2 - x'_2|) = 1\}\right| n^{d-2} \geq$$

$$\geq \left(\sum_{i \leq n/2} \varphi(i) \right) n^{d-2} = \left(\frac{3}{\pi^2} \left(\frac{n}{2}\right)^2 + O\left(\frac{n}{2} \ln \frac{n}{2}\right) \right) n^{d-2} = \Omega(n^d).$$

Since $|E_n^d| = n^d$, this lower bound matches a trivial upper bound, and therefore $|S(f,\mathfrak{T}(d,n,k))| = \Theta(n^d)$. ∎

3.2. Teaching sets of functions in $\mathfrak{T}(2,n,*)$

In the previous section we proved that for a function from $\mathfrak{T}(d,n,*), d \geq 2$ the set of its essential points is also the unique minimal teaching set. In this section we consider the class $\mathfrak{T}(2,n,*)$ and describe the structure of the set of essential points for a function in this class. We also give an upper bound on the cardinality of this set.

Let us consider an arbitrary function $f \in \mathfrak{T}(2,n,*)$. Note that $P(f)$ can be the empty set, a point, a segment or a polygon. Let $P(f)$ be a segment or a polygon, that is $|M_1(f)| > 1$, and let $a_1x_1 + a_2x_2 = a_0$ be the edge equality for an edge e of $P(f)$. Without loss of generality we may assume that $\mathrm{GCD}(a_1, a_2) = 1$. Denote by *edge inequality* for edge e inequality $a_1x_1 + a_2x_2 \leq a_0$ or/and $a_1x_1 + a_2x_2 \geq a_0$ if it is true for all points of $P(f)$. Note that if $P(f)$ is a segment, then it has one edge but two edge inequalities corresponding to the edge. If $P(f)$ is a polygon, then it has exactly one edge inequality for each edge. Hence the number of edge inequalities for $P(f)$ is equal to the number of its vertices.

Let f be a function from $\mathfrak{T}(2,n,*)$ with $|M_1(f)| > 1$ and let

$$a_{i1}x_1 + a_{i2}x_2 \leq a_{i0}, \quad i = 1, \ldots, |\mathrm{Vert}(P(f))|$$

be edge inequalities for $P(f)$. The *extended edge inequality* for an edge e of $P(f)$ is $a_1x_1 + a_2x_2 \leq a_0 + 1$, where $a_1x_1 + a_2x_2 \leq a_0$ is the corresponding edge inequality for e. By $P'(f)$ we denote the following extension of $P(f)$

$$\{x = (x_1, x_2) : a_{i1}x_1 + a_{i2}x_2 \leq a_{i0} + 1, \quad i = 1, \dots, |\mathrm{Vert}(P(f))|\}. \tag{2}$$

We also let

$$\Delta P(f) = P'(f) \setminus P(f).$$

It follows from the definition that $P'(f)$ contains $P(f)$, and for every straight line l' containing an edge of $P'(f)$ there exists an edge in $P(f)$ belonging to the closest parallel to the l' straight line which contains integer points.

If P is a polygon then denote by $\mathcal{P}(P)$ the perimeter of P, by $\mathcal{S}(P)$ the area of P and by $q_{min}(P)$ the minimum angle between neighboring edges of P.

The next proposition uses the Pick's formula (see [10]) for the area of a convex polygon P with integer vertices:

$$\mathcal{S}(P) = \mathrm{Int}(P) + \frac{B(P)}{2} - 1.$$

Proposition 10. *Let $f \in \mathfrak{T}(2, n, *)$ and $\mathcal{S}(P(f)) > 0$. Then $D(f) = \Delta P(f) \cap M_0(f)$.*

Proof. Note that by construction all integer points of $\Delta P(f)$ lie on the border of $P'(f)$, which implies that $\Delta P(f) \cap M_0(f) \subseteq D(f)$. Consider $x = (x_1, x_2)$ such that $|\mathrm{Conv}(P(f)) \cup \{x\}) \cap M_0(f)| = 1$. To show that $x \in \Delta P(f)$ it is sufficient to prove that x is a solution of the system of inequalities (1), that is each extended edge inequality for $P(f)$ holds true for x. Obviously, if an edge inequality is true for x, then the corresponding extended edge inequality is also true. Let e be the edge whose edge inequality is false for x, that is $a_1x_1 + a_2x_2 > a_0$. All integer points of the triangle $Tr = \mathrm{Conv}(e \cup \{x\})$ belongs to $e \cup \{x\}$. Since Tr has integer vertices, its area can be calculated by the Pick's formula:

$$\mathcal{S}(Tr) = \frac{|(e \cup \{x\}) \cap E_n^2|}{2} - 1 = \frac{|e \cap E_n^2| - 1}{2}.$$

Comparing resulting equation with the classical triangle area formula $\mathcal{S}(Tr) = \frac{l(e)h_x}{2}$ we conclude that

$$h_x = \frac{|e \cap E_n^2| - 1}{l(e)},$$

where h_x is the distance between point x and the line containing e.

Now consider an integer point $y = (y_1, y_2)$ for which $a_1y_1 + a_2y_2 = a_0 + 1$. Using the same arguments it is easy to show that

$$h_y = \frac{|e \cap E_n^2| - 1}{l(e)}.$$

Hence, x and all integer points of the line $a_1y_1 + a_2y_2 = a_0 + 1$ have the same distance to the line containing e. It means that $a_1x_1 + a_2x_2 = a_0 + 1$, that is the extended edge inequality for e is true for x and x belongs to $P(f)$, therefore $D(f) \subseteq \Delta P(f) \cap M_0(f)$. ∎

Corollary 11. *Let $f \in \mathfrak{T}(2, n, *)$ and $\mathcal{S}(P(f)) > 0$. Then*

$$S(f, \mathfrak{T}(2, n, *)) = (\Delta P(f) \cap M_0(f)) \cup \mathrm{Vert}(P(f)).$$

The next lemma establishes relationship between the perimeters of $P(f)$ and $P'(f)$ to help us to estimate the cardinality of the set of essential points of a function from $\mathfrak{T}(2, n, *)$.

Lemma 12. *Let $f \in \mathfrak{T}(2, n, *)$ and $\mathcal{S}(P(f)) > 0$. Then*

$$\mathcal{P}(P'(f)) \leq \mathcal{P}(P(f)) + 2 \sum_{i=1}^{|\mathrm{Vert}(P(f))|} \cot \frac{q_i(P(f))}{2},$$

where $q_i(P(f))$ for $i \in \{1, \dots, |\mathrm{Vert}(P(f))|\}$ are the angles between neighboring edges of $P(f)$.

Proof. Denote by P'' the set of points satisfying such a condition that if an edge inequality is false for a point, then the distance between the point and the straight line containing the corresponding edge is at most 1. Note that points of $P'(f)$ also satisfy the specified condition, so $P'(f) \subseteq P''$ and, consequently, $\mathcal{P}(P'(f)) \leq \mathcal{P}(P'')$ (see Fig. 3). Further, P'' is a convex polygon with $|\mathrm{Vert}(P(f))|$ vertices, and each edge e'' of P'' is parallel to some edge e of $P(f)$ and is at distance 1 from the line containing e. Let q_i, q_{i+1} for some $i \in \{1, \dots, |\mathrm{Vert}(P(f))| - 1\}$ be the angles between e and its neighboring edges in $P(f)$. By construction of P'' we have:

$$l(e'') = l(e) + \cot \frac{q_i}{2} + \cot \frac{q_{i+1}}{2}.$$

Summing up the lengths of all edges of P'' we have:

$$\mathcal{P}(P'(f)) \leq \mathcal{P}(P'') = \mathcal{P}(P(f)) + 2 \sum_{i=1}^{|\mathrm{Vert}(P(f))|} \cot \frac{q_i(P(f))}{2}.$$

■

Theorem 13. *Let $f \in \mathfrak{T}(2, n, *)$ and $\mathcal{S}(P(f)) > 0$. Then*

$$|S(f, \mathfrak{T}(2, n, *))| = O\left(\min\left(n, \mathcal{P}(P(f)) + \frac{1}{q_{min}(P(f))}\right)\right).$$

Proof.

By Corollary 11 we have $S(f, \mathfrak{T}(2, n, *)) = (\Delta P(f) \cap M_0(f)) \cup \mathrm{Vert}(P(f))$. Since every point of $S(f, \mathfrak{T}(2, n, *))$ is integer and either belongs to the border of $P(f)$ or to the border of $P'(f)$, the cardinality of $S(f, \mathfrak{T}(2, n, *))$ can be bounded from above by the sum of the perimeters $\mathcal{P}(P(f))$ and $\mathcal{P}(P'(f))$. So we have:

$$|S(f, \mathfrak{T}(2, n, *))| \leq \mathcal{P}(P(f)) + \mathcal{P}(P'(f)) \leq 2\mathcal{P}(P(f)) + \sum_{i=1}^{|\mathrm{Vert}(P(f))|} 2 \cot \frac{q_i(P(f))}{2},$$

where q_i for $i \in \{1, \dots, |\mathrm{Vert}(P(f))|\}$ are the angles between neighboring edges of $P(f)$.

As the number of integer vertices of a convex polygon is not more than the perimeter of the polygon, we have $|\mathrm{Vert}(P(f))| \le \mathcal{P}(P(f))$. Obviously, only 2 angles of a convex polygon can be less than $\frac{\pi}{3}$. Therefore

$$2\mathcal{P}(P(f)) + \sum_{i=1}^{|\mathrm{Vert}(P(f))|} 2\cot\frac{q_i(P(f))}{2} \le$$

$$\le 2\mathcal{P}(P(f)) + 4\cot\frac{q_{min(P(f))}}{2} + \sqrt{3}(\mathcal{P}(P(f)) - 2).$$

Since $0 \le \frac{q_{min}(P(f))}{2} < \frac{\pi}{2}$, we can conclude:

$$\cot\frac{q_{min}(P(f))}{2} \le \frac{1}{\sin\frac{q_{min(P(f))}}{2}} = O\left(\frac{1}{q_{min}(P(f))}\right).$$

Therefore

$$|S(f, \mathfrak{T}(2, n, *))| = O\left(\mathcal{P}(P(f)) + \frac{1}{q_{min}(P(f))}\right).$$

Finally, since

$$\mathcal{P}(P(f)) + \mathcal{P}(P'(f)) = O(n),$$

we conclude:

$$|S(f, \mathfrak{T}(2, n, *))| = O\left(\min\left(n, \mathcal{P}(P(f)) + \frac{1}{q_{min}(P(f))}\right)\right).$$

■

Example 14. *Consider a function $f \in \mathfrak{T}(2, 12, *)$ (see Fig. 4). The gray set is $\Delta P(f)$. The black stars are the points from* $\mathrm{Vert}(P(f))$ *and the white stars are the points from* $\Delta P(f) \cap M_0(f)$.

Proposition 15. *Let $f \in \mathfrak{T}(2, n, *)$ and $M_1(f) > 1$. Then f is a $|\mathrm{Vert}(P(f))|$-threshold function and the sets of essential points of f with respect to $\mathfrak{T}(2, n, *)$ and with respect to $\mathfrak{T}(2, n, |\mathrm{Vert}(P(f))| + 1)$ coincide.*

Proof. Lemma 6 shows that functions f with $|\mathrm{Vert}(P(f))| = 2$ are 2-threshold. Let $|\mathrm{Vert}(P(f))| > 2$. The polygon $P(f)$ is a solution of a system of $|\mathrm{Vert}(P(f))|$ inequalities, and therefore f is a $|\mathrm{Vert}(P(f))|$-threshold. For any $x \in \mathrm{Vert}(P(f))$ we can add one inequality to the system (1) to get a function $f' \in \mathfrak{T}(2, n, |\mathrm{Vert}(P(f))| + 1)$ such that $M_1(f') = M_1(f) \setminus \{x\}$, hence $\mathrm{Vert}(P(f)) \subseteq S(f, \mathfrak{T}(2, n, |\mathrm{Vert}(P(f))| + 1)$.

Now, consider an arbitrary point $x \in D(f)$ and a function $f' \in \mathfrak{T}(2, n, *)$ with $M_1(f) = \mathrm{Conv}(P(f) \cup \{x\}) \cap E_n^2$. Obviously, $|\mathrm{Vert}(P(f'))| \le |\mathrm{Vert}(P(f))| + 1$ and f' is a $(|\mathrm{Vert}(P(f))| + 1)$-threshold function. The functions f and f' differ in the unique point x and belong to the classes of $|\mathrm{Vert}(P(f'))|$-threshold and $(|\mathrm{Vert}(P(f))| + 1)$-threshold functions, respectively. Therefore $x \in S(f, \mathfrak{T}(2, n, |\mathrm{Vert}(P(f))| + 1))$ and $D(f) \subseteq S(f, \mathfrak{T}(2, n, |\mathrm{Vert}(P(f))| + 1))$. According to Theorem 8 the sets of essential points of f with respect to $\mathfrak{T}(2, n, *)$ and with respect to $\mathfrak{T}(2, n, |\mathrm{Vert}(P(f))| + 1)$ coincide. ■

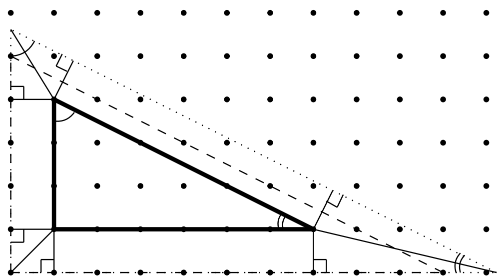

Figure 3: $P(f)$ (bold solid triangle), $P'(f)$ (dashed triangle) and P'' (dotted triangle).

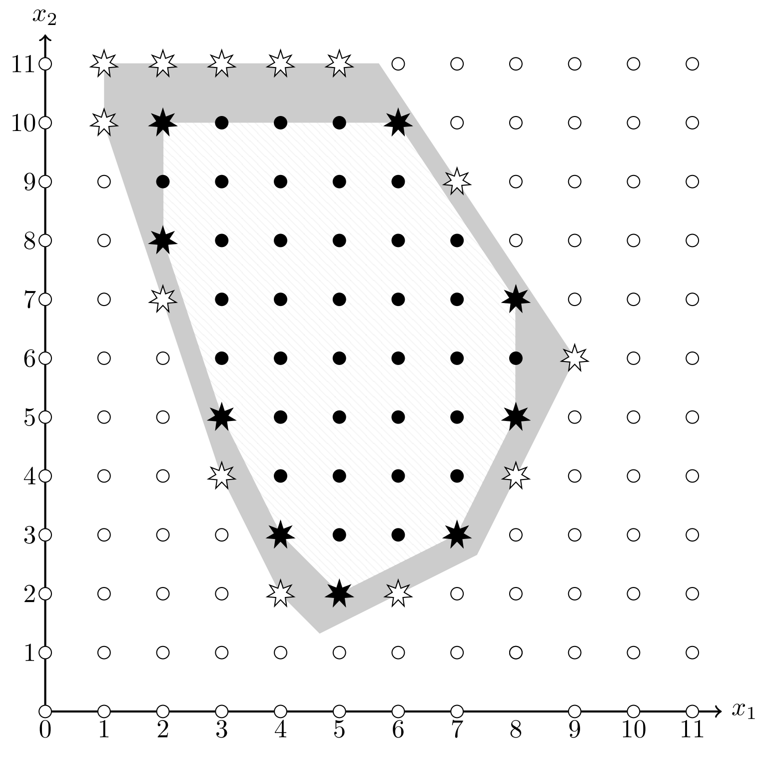

Figure 4: The gray set is $\Delta P(f)$, the stripped area is $P(f)$, and the union of both of them is $P'(f)$.

Example 16. *Consider a function $f \in \mathfrak{T}(2,n,4)$ with $M_1(f) = \{(1,1),(1,2),(2,1)\}$ (see Fig. 5). We have* $\mathrm{Vert}(P(f)) = \{(1,1),(1,2),(2,1)\}$ *and f is a 3-threshold function. Further,* $\Delta P(f) \cap E_4^2 = \{(0,0),(1,0),(2,0),(3,0),(0,1),(3,1),(0,2),(2,2),(3,2),(0,3),(1,3)\}$, *and hence* $S(f,\mathfrak{T}(2,n,*)) = E_4^2 \setminus \{(3,2),(2,3),(3,3)\} = S(f,\mathfrak{T}(2,n,4))$.

3.3. The teaching set of functions from $\mathfrak{T}(2,n,2)$ with a unique defining set of threshold functions

In this section we consider the subset of 2-threshold functions over E_n^2, for which the cardinality of minimal teaching set can be bounded by a constant. Also we show that for such 2-threshold functions the number of minimal teaching sets can grow as $\Omega(n^2)$.

Let $f \in \mathfrak{T}(2,n)$ and let a_0, a_1, a_2 be real numbers which are not all zero. We call the line $a_1x_1 + a_2x_2 = a_0$ an *i-separation line* (or just *separation line*) of f if there exists $i \in \{0,1\}$ such that

$$x = (x_1, x_2) \in M_i(f) \Longleftrightarrow a_1x_1 + a_2x_2 \leq a_0.$$

For example, the equality corresponding to a threshold inequality of f defines a 1-separation line of f. Let us prove some properties of separation lines of threshold functions.

It is known [11] that $|S(g)| \in \{3,4\}$ and $|S_1(g)|, |S_0(g)| \in \{1,2\}$ for any $g \in \mathfrak{T}(2,n)$ and the 1-valued essential points of g are adjacent vertices of $P(g)$.

Proposition 17. *Let $f \in \mathfrak{T}(2,n)$. For any $i \in \{0,1\}$ there exists an i-separation line of f which contains all points of $S_i(f)$.*

Proof. Clearly, it is enough to prove the proposition for $i = 1$. Denote by l some 1-separation line of f which does not contain integer points and let $x \in S_1(f)$. There exists a function $g \in \mathfrak{T}(2,n)$ such that x distinguishes f from g, that is $f(y) = g(y)$ for all $y \in E_n^2 \setminus \{x\}$ and $g(x) = 0$. Denote by l' some 1-separation line for g which also does not contain integer points. If l and l' are parallel lines then x lies between them. In this case we can pass through x a parallel to l and l' straight line l'' which will be a 1-separate line of f. If l and l' intersect in some point y, then the straight line l'' which intersects x and y is a 1-separation line of f. Thus, for any essential point there exists a separation line which intersects x and does not contain any other integer points. This proves the proposition for $|S_1(f)| = 1$.

Now let $|S_1(f)| = 2$ and $S_1(f) = \{x,y\}$. There exist functions $g_x, g_y \in \mathfrak{T}(2,n)$ such that $f(z) = g_j(z)$ for all $z \in E_n^2 \setminus \{j\}$ and $g_j(j) = 0$, where $j \in \{x,y\}$. Denote by l_j a 1-separation line for g_j which does not contain integer points except point j. By construction, sets $M_0(g_x) \cap M_0(g_y)$ and $M_1(g_x) \cap M_1(g_y)$ are separated by the straight line l' containing x and y. Since $M_0(g_x) \cap M_0(g_y) = M_0(f)$ and $M_1(g_x) \cap M_1(g_y) = M_1(f) \setminus l'$, the line l' is a 1-separation line. ■

Proposition 18. *Let $f \in \mathfrak{T}(2,n)$ and l is an i-separation line for f for some $i \in \{0,1\}$. Then* $\mathrm{Vert}(\mathrm{Conv}(l \cap E_n^2)) \subseteq S_i(f)$.

Proof. Assume without loss of generality that $i = 1$ and l is a 1-separation line. If $l \cap E_n^2 = \emptyset$, then the proposition is obvious. Suppose $l \cap E_n^2 = \{x\}$, that is l intersects E_n^2

in exactly one point x. It is easy to see that l is also a 0-separation line for a function $g \in \mathfrak{T}(2,n)$ which coincides with f on $E_n^2 \setminus \{x\}$ and $g(x) = 0$, therefore x is an essential point for f. Since l is a 1-separation line for f and $x \in l$, we conclude that $x \in S_1(f)$.

Now suppose that $|l \cap E_n^2| > 1$ and $\mathrm{Vert}(\mathrm{Conv}(l \cap E_n^2)) = \{x, y\}$. We can turn l around x on a small angle (to not intersect any other integer points) in such a direction that y would be on the same halfspace from the line as other points of $M_1(f)$. New line l' will be 1-separation line for f containing exactly one integer point x, and, as we showed above, $x \in S_1(f)$. The same arguments are true for y, that is $y \in S_1(f)$. ∎

Proposition 19. *Let $f \in \mathfrak{T}(2,n)$. The sets $S_0(f)$ and $S_1(f)$ belong to the parallel separation lines and there is no integer points between the lines.*

Proof. Assume without loss of generality that $|S_1(f)| = 2$ and $S_1(f) = \{x, y\}$. By proposition 17 the line l containing $S_1(f)$ is a 1-separation line for f. We can make a translation of l in direction to $M_0(f)$ to the closest line l' which intersects at least one point from $M_0(f)$. If $|S_0(f)| = 1$ and $S_0(f) = \{z\}$, then $z \in l'$ and the proposition holds. Let $|S_0(f)| = 2$ and $S_0(f) = \{z, u\}$. Note that triangles $\triangle xyz$ and $\triangle xyu$ contain no other integer points, except the vertices and the points on the segment xy. By the Pick's formula both triangles have the same area. It means that both z and u are at the same distance from l and lie on the line l'. ∎

Theorem 20. *Let $f \in \mathfrak{T}(2,n,2)$ and $M_1(f) \cap B(\mathrm{Conv}(E_n^2)) \neq \emptyset$, and let some set of threshold functions $\{f_1, f_2\}$ defining f satisfies the following system:*

$$\begin{cases} S(f_1) \cap M_0(f_2) = \emptyset; \\ S(f_2) \cap M_0(f_1) = \emptyset. \end{cases} \tag{3}$$

Then $\{f_1, f_2\}$ is a unique defining set of f and

$$\sigma(f, \mathfrak{T}(2,n,2)) \leq 9.$$

Proof. Note that

$$B(\mathrm{Conv}(E_n^2)) = \{x \in E_n^2 : x_1 = 0 \vee x_2 = 0 \vee x_1 = n-1 \vee x_2 = n-1\}.$$

We consider two cases depending on the cardinalities of $S_0(f_1)$, $S_0(f_2)$.

Let $|S_0(f_i)| = 1$ for some $i \in \{1, 2\}$. Assume, without loss of generality, that $|S_0(f_1)| = 1$, that is $S_0(f_1) = \{u\}$. Then $|S_1(f_1)| = 2$ and $S_1(f_1) = \{v_1, v_2\}$. Consider an arbitrary function $f' \in \mathfrak{T}(2,n,2)$ which agrees with f on $S(f_1) \cup S(f_2)$ and some of its defining set of threshold functions $F' = \{f'_1, f'_2\}$. From the first equation of the system (3) it follows that $f_1(x) = f(x) = f'(x)$ for every $x \in S(f_1)$. Hence one of the functions from F', say f'_1, should take the value 0 on u and the value 1 on v_1 and v_2, therefore

$$f'_1 = f_1. \tag{4}$$

From the second equation of the system (3) we have $f_2(x) = f(x) = f'(x)$ for every $x \in S(f_2)$. This together with (4) imply that f'_2 agrees with f_2 on $S(f_2)$, and therefore $f'_2 = f_2$. We showed that $\{f_1, f_2\} = F'$, and hence f' coincides with f and $\{f_1, f_2\}$ is a unique defining set for f. Moreover, $S(f_1) \cup S(f_2)$ is a teaching set of f and $|S(f_1) \cup S(f_2)| \leq 7$.

Now suppose that $|S_0(f_1)| = |S_0(f_2)| = 2$, that is $S_0(f_1) = \{u_1, u_2\}$, $S_0(f_2) = \{v_1, v_2\}$. Denote by $G \subseteq \mathfrak{T}(2, n, 2)$ a set of 2-threshold functions, which agree with f on $S(f_1) \cup S(f_2)$. From the conditions of the theorem it follows that $S_0(f_i) \in M_1(f_j)$ for $i \neq j$. Note that $S_0(f_1) \cup S_0(f_2)$ is a set of vertices of a convex quadrilateral $P = (u_1, u_2, v_1, v_2)$, and for each of the threshold functions $\{f_1, f_2\}$ vertices from its teaching set are neighboring (see Fig. 6). This implies that G is the union of two sets:

$$G_1 = \{g \,|\, g \in G, \exists g_1, g_2 \in \mathfrak{T}(2, n) : g = g_1 \wedge g_2, \{u_1, u_2\} \subseteq M_0(g_1), \text{ and } \{v_1, v_2\} \subseteq M_0(g_2)\}$$

and

$$G_2 = \{g \,|\, g \in G, \exists g_1, g_2 \in \mathfrak{T}(2, n) : g = g_1 \wedge g_2, \{u_1, v_2\} \subseteq M_0(g_1), \text{ and } \{u_2, v_1\} \subseteq M_0(g_2)\}.$$

Applying the same arguments as in the previous case where $|S_0(f_i)| = 1$ it can be shown that $G_1 = \{f\}$. Now, to prove the uniqueness of a defining set of f it is sufficient to demonstrate that $f \notin G_2$. To this end, we first show that

$$M_1(f) \cap \bigcup_{g \in G_2} M_1(g) \subseteq \mathrm{Int}(P). \tag{5}$$

By Proposition 17 the line l containing u_1, u_2 and the line l' containing v_1, v_2 are a 0-separation lines of f, and hence all points of $M_1(f)$ lie between l and l'. Note that for every threshold function h the sets $\mathrm{Conv}(M_1(h))$ and $\mathrm{Conv}(M_0(h))$ do not intersect. These facts imply that $M_1(f) \cap M_1(g)$ should lie between l and l' and between the segments u_1v_2 and u_2v_1, which proves (5). Now it follows from (5) and the condition of the theorem that $f \notin G_2$ and $\{f_1, f_2\}$ is a unique defining set of f.

Finally, we are interested in a point $x' \in E_n^2$ which would distinguish f from every function in G_2, that is $f(x') \neq g(x')$ for all $g \in G_2$. By (5) we have $g(x) = 0$ for all $g \in G_2$ and $x \in M_1(f) \setminus \mathrm{Int}(P)$. Since $B(\mathrm{Conv}(E_n^2)) \cap \mathrm{Int}(P) = \emptyset \neq B(\mathrm{Conv}(E_n^2)) \cap M_1(f)$, we can take an arbitrary point from $B(\mathrm{Conv}(E_n^2)) \cap M_1(f)$ as x' and obtain a teaching set T for f which is equal to $S(f_1) \cup S(f_2) \cup \{x'\}$. Note that any such a teaching set T is minimal and $|T| \leq 9$.

■

Remark 21. *Theorem 20 also holds when the domain is a convex subset of* E_n^2.

Corollary 22. *Let* $f \in \mathfrak{T}(2, n, 2)$ *and there is a unique set of threshold functions* $\{f_1, f_2\}$ *defining* f*. If* $M_1(f) \cap B(\mathrm{Conv}(E_n^2)) \neq \emptyset$*, then*

$$\sigma(f, \mathfrak{T}(2, n, 2)) \leq 9.$$

Proof. By Proposition 3, for f_1 and f_2 the following is true:

$$\begin{cases} S(f_1) \cap M_0(f_2) = \emptyset; \\ S(f_2) \cap M_0(f_1) = \emptyset. \end{cases}$$

Therefore f satisfies the conditions of Theorem 20. ■

Recall that $J(f, C)$ denotes the number of minimal teaching sets of a function f with respect to a class C. Using the set of functions G_2 from Theorem 20 the next lemma proves that number of minimal teaching sets of 2-threshold functions can grow as $\Omega(n^2)$.

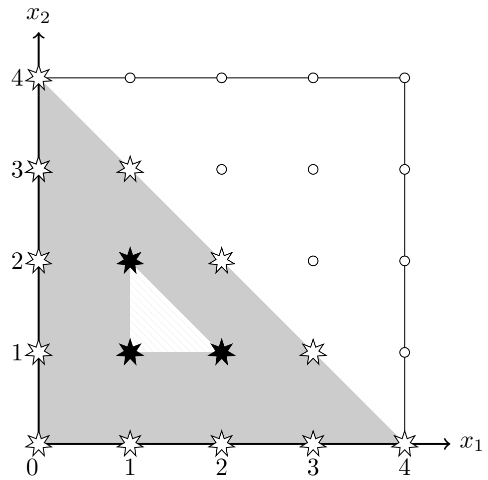

Figure 5: The gray shape is $\Delta P(f)$, the stripped area is $P(f)$.

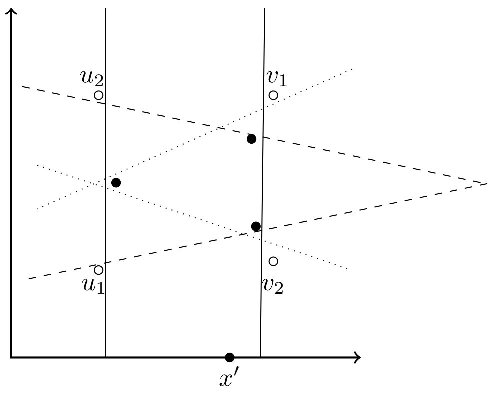

Figure 6: The dashed and dotted lines correspond to pair of different functions from G_2, and the solid lines correspond to the function f.

Lemma 23.

$$\max_{f \in \mathfrak{T}(2,n,2)} J(f, \mathfrak{T}(2,n,2)) = \Omega(n^2).$$

Proof. Let

$$m = m(n) = \left\lfloor \frac{n-1}{4} \right\rfloor .$$

For $n \geq 21$ let $f^{(n)} \in \mathfrak{T}(2,n,2)$ be defined by threshold functions $f_1^{(n)}$ and $f_2^{(n)}$ with the corresponding inequalities:

$$\begin{cases} -3x_1 - 4x_2 \leq -25, \\ 3x_1 + 4x_2 \leq 12m - 1. \end{cases}$$

Note that $l : 3x_1 + 4x_2 = 12m - 1$ is a 1-separation line of $f_2^{(n)}$ and by Proposition 18 we have $\mathrm{Vert}(\mathrm{Conv}(l \cap E_n^2)) \subseteq S_1(f_2^{(n)})$. By construction, l is not parallel to x_2-axis and contains at least two integer points from E_n^2. Therefore $\mathrm{Conv}(l \cap E_n^2)$ is a segment and its vertices are solutions of the following two integer linear programming problems with constraints $n \in \mathbb{Z}$, $n \geq 21$ and $m = \lfloor \frac{n-1}{4} \rfloor$:

$$\begin{cases} \max x_1, \\ 3x_1 + 4x_2 = 12m - 1, \\ 0 \leq x_1 \leq n - 1, \\ 0 \leq x_2 \leq n - 1, \\ x_1, x_2 \in \mathbb{Z}, \end{cases} \qquad \begin{cases} \min x_1, \\ 3x_1 + 4x_2 = 12m - 1, \\ 0 \leq x_1 \leq n - 1, \\ 0 \leq x_2 \leq n - 1, \\ x_1, x_2 \in \mathbb{Z}. \end{cases}$$

It is easy to check that the solutions of the above problems are $(4m-3, 2)$ and $(1, 3m-1)$, therefore $S_1(f_2^{(n)}) = \{(4m-3, 2), (1, 3m-1)\}$.

Now, the closest parallel to l line, which contains 0-values of f, is $l' : 3x_1 + 4x_2 = 12m$. By Proposition 19 all points of $S_0(f_2^{(n)})$ are vertices of $\mathrm{Conv}(l' \cap E_n^2)$, and to find $S_0(f_2^{(n)})$ we can use the same arguments as we did for $S_1(f_2^{(n)})$. The same is true for $f_1^{(n)}$ and the set $S(f_1^{(n)})$, hence the final conclusion looks as follows:

$$\begin{cases} S_0(f_1^{(n)}) = \{u_1 = (8,0), u_2 = (0,6)\}, \\ S_1(f_1^{(n)}) = \{u_3 = (7,1), u_4 = (3,4)\}, \\ S_0(f_2^{(n)}) = \{v_1 = (0,3m), v_2 = (4m,0)\}, \\ S_1(f_2^{(n)}) = \{v_3 = (4m-3,2), v_4 = (1,3m-1)\}. \end{cases}$$

Note that $f^{(n)}$ satisfies the conditions of Corollary 22 and Theorem 20 and quadrilateral P from the proof of Theorem 20 has vertices u_1, u_2, v_1, and v_2. Denote by $G \subseteq \mathfrak{T}(2,n,2)$ the set of functions such that for every $g \in G$ and for some threshold functions g_1, g_2 defining g the following is true:

$$S_1(f_1^{(n)}) \cup S_1(f_2^{(n)}) \subseteq M_1(g),$$

$$\{u_1, v_2\} \subset M_0(g_1),$$

$$\{u_2, v_1\} \subset M_0(g_2).$$

The set G corresponds to the set G_2 from the proof of Theorem 20, therefore all functions from G and only them agree with $f^{(n)}$ on $S(f_1^{(n)}) \cup S(f_2^{(n)})$. Let us bound from below the number of points x' such that

$$f^{(n)}(x') \neq g(x') \text{ for all } g \in G. \tag{6}$$

Denote by $R(n)$ the triangle with vertices v_3, v_4 and $(n-1, n-1)$ and by $L(n)$ the segment $v_3 v_4$. It is clear that $R(n) \cap M_1(f^{(n)}) = L(n) \cap E_n^2$. By construction of the set G, the inclusion $R(n) \cap E_n^2 \subset M_1(g)$ holds for any $g \in G$. It means that any point from $(R(n) \setminus L(n)) \cap E_n^2$ distinguishes $f^{(n)}$ from any function in G. Therefore the number of minimal teaching sets for $f^{(n)}$ can be lower bounded by the cardinality of $(R(n) \setminus L(n)) \cap E_n^2$, which is equal to $|R(n) \cap E_n^2| - |L(n) \cap E_n^2|$.

The number of integer points in $L(n)$ can be calculated through the GCD of the differences between coordinates of v_3 and v_4:

$$|L(n) \cap E_n^2| = 2 + \text{GCD}((4m-3)-1, (3m-1)-2) = m+1.$$

The number of integer points in $R(n)$ can be calculated by means of the Pick's formula. Indeed, since $R(n)$ is a triangle with integer vertices, we have

$$\mathcal{S}(R(n)) = |\text{Int}(R(n))| + \frac{|B(R(n))|}{2} - 1$$

and therefore

$$|R(n) \cap E_n^2| = |\text{Int}(R(n))| + |B(R(n))| = \mathcal{S}(R(n)) + \frac{|B(R(n))|}{2} + 1.$$

Now since

$$|B(R(n))| \geq |L(n) \cap E_n^2| + 1 \geq m + 2$$

and

$$\mathcal{S}(R(n)) = \frac{|(m-1)(12m-7n+6)|}{2},$$

we conclude that

$$|(R(n) \setminus L(n)) \cap E_n^2| \geq \frac{|(m-1)(12m-7n+6)|}{2} + \frac{m+2}{2} + 1 - (m+1) = \Theta(n^2).$$

That is the number of minimal teaching sets for the function $f^{(n)}$ grows as $\Omega(n^2)$.

■

4. Open problems

In this paper, we investigated structural and quantitative properties of sets of essential points and minimal teaching sets of k-threshold functions.

We proved that a function in the class $\mathfrak{T}(d, n, *)$ has a unique minimal teaching set which is equal to the set of essential points of this function with respect to the class. For a function in the class $\mathfrak{T}(2, n, *)$ we estimated the cardinality of the set of essential points

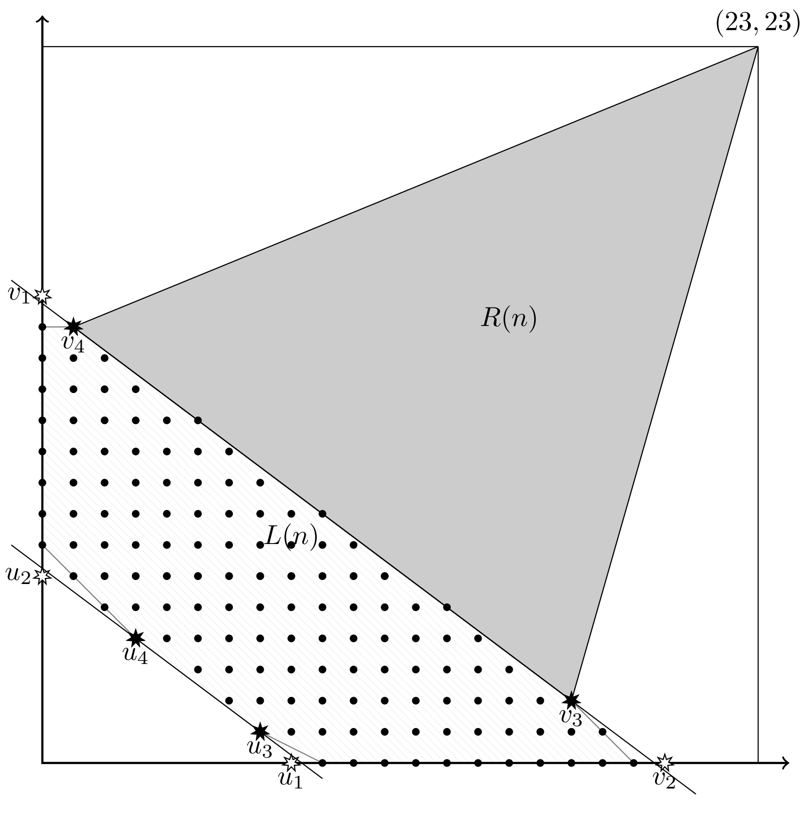

Figure 7: An example of $f^{(24)}$, the black points are the points of $M_1(f)$, the gray shape is $R(n)$, the stripped area is $P(f)$.

of the function. It would be interesting to find analogous bounds on the cardinality of the set of essential points of a function in $\mathfrak{T}(d, n, *)$ for $d > 2$.

We considered $\mathfrak{T}(2, n, 2)$ and proved that the set of essential points of a function in this class is not necessary a minimal teaching set. Moreover we showed that $J(\mathfrak{T}(2, n, 2)) = \Omega(n^2)$. Also in the class $\mathfrak{T}(2, n, 2)$ we identified functions with minimal teaching sets of cardinality at most 9. It would be interesting to estimate the proportion of functions with this property in the class $\mathfrak{T}(2, n, 2)$.

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